

# Harmonic Spinors for Twisted Dirac Operators

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## Abstract

We show that for a suitable class of “Dirac-like” operators there holds a Gluing Theorem for connected sums. More precisely, if  $M_1$  and  $M_2$  are closed Riemannian manifolds of dimension  $n \geq 3$  together with such operators, then the connected sum  $M_1 \# M_2$  can be given a Riemannian metric such that the spectrum of its associated operator is close to the disjoint union of the spectra of the two original operators. As an application, we show that in dimension  $n \equiv 3 \pmod{4}$  harmonic spinors for the Dirac operator of a spin,  $\text{spin}^c$ , or  $\text{spin}^h$  manifold are not topologically obstructed.

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## 1 Introduction

The interplay between geometric, topological, and analytic invariants of Riemannian manifolds is one of the major topics of current investigations in global analysis and differential geometry. A classical example is provided by Hodge-deRham theory. The dimension of the kernel of the Laplace-Beltrami operator acting on  $p$ -forms on a closed Riemannian manifold is an important topological invariant, the  $p$ -th Betti number. In particular, it does not depend on the Riemannian metric. The question arises whether one can obtain further topological invariants using other natural operators like the Dirac operator on a closed Riemannian spin manifold.

There are topological restrictions against existence of harmonic spinors in dimension 2. A 2-sphere equipped with an arbitrary Riemannian metric  $g$  does not have non-trivial harmonic spinors. This can be deduced e.g. from the eigenvalue estimate [3, Thm. 2]

$$\lambda^2 \geq \frac{4\pi}{\text{area}(S^2, g)}$$

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which holds for all Dirac eigenvalues  $\lambda$  on  $(S^2, g)$ . At the moment the 2-sphere is the only closed spin manifold for which one knows non-existence of harmonic spinors for *all* Riemannian metrics and all spin structures. In fact, the present paper gives some evidence to the conjecture that  $S^2$  is the only such manifold.

On a surface of genus 1 or 2 the dimension of the space of harmonic spinors does not depend on the Riemannian metric, but it does depend on the choice of spin structure. For genus larger than 2 it depends on both, the Riemannian metric and the spin structure. One can always choose the metric and the spin structure in such a way that there are non-trivial harmonic spinors [11]. For hyperelliptic surfaces one can compute the dimension for all spin structures [7].

Hitchin [11] computed the Dirac spectrum for a suitable 1-parameter family of metrics on  $S^3$ . It turns out that for generic parameter values there are no non-trivial harmonic spinors but for special choices of the parameter the dimension of the space of harmonic spinors becomes arbitrarily large.

If the dimension of the manifold is divisible by 4, then the Atiyah-Singer Index Theorem [2, Thm. 5.3] implies

$$\dim\{\text{harmonic spinors}\} \geq |\hat{A}(M)|$$

where  $\hat{A}(M)$  is a topological invariant, the  $\hat{A}$ -genus of  $M$ . Kotschick [12] exhibited algebraic surfaces for which  $\dim\{\text{harmonic spinors}\}$  exceeds  $|\hat{A}(M)|$  arbitrarily much.

All known examples indicate that the following conjecture should hold.

**Conjecture.** *Let  $M$  be a closed spin manifold of dimension  $n \geq 3$ . Let the spin structure on  $M$  be fixed.*

*Then there exists a Riemannian metric on  $M$  such that there are non-trivial harmonic spinors.*

Hitchin [11] proved this conjecture by differential topological methods in dimension  $n \equiv 0, \pm 1 \pmod{8}$ . In [5] we showed by analytic methods that the conjecture also holds in dimension  $n \equiv 3 \pmod{4}$ .

In this paper which should be regarded as a sequel to [5] we enlarge the class of operators for which the analogous statement holds. This is possible because the methods of [5] are essentially local. This class of operators contains in particular the Dirac operators of  $\text{spin}^c$  or  $\text{spin}^h$  manifolds which have regained interest recently because of Seiberg-Witten theory [17], see [14, 16] for an introduction. Some care has to be taken with the statement of the result. In contrast to spin manifolds the Dirac operator of a  $\text{spin}^c$  manifold is not determined by the Riemannian metric alone but also depends on the choice of a connection on the canonical line bundle. A similar remark holds for  $\text{spin}^h$  manifolds. The nature

of the Seiberg-Witten equations indicates that one should look at modifications of the metric and of the connection on the canonical line bundle separately. It turns out that a modification of the metric is sufficient to produce a non-trivial kernel. Theorem 4.1 applied to  $\text{spin}^c$  manifolds says

**Theorem.** *Let  $M$  be a closed  $\text{spin}^c$  manifold of dimension  $n \equiv 3 \pmod{4}$ . Let a connection on the canonical line bundle be fixed. Then there exists a Riemannian metric on  $M$  such that there are non-trivial harmonic spinors for the associated Dirac operator.*

For  $n = 3$  this can be rephrased in terms of Seiberg-Witten equations for 3-dimensional (!) manifolds. Of course, it would be interesting to have the analogous statement in dimension 4, but at the moment this seems out of reach.

The construction shows that this metric can be obtained by deforming any given metric in an arbitrarily small open set while keeping it unchanged outside. Since the construction is local Theorem 4.1 applies to self-adjoint elliptic operators which look like a twisted Dirac operator in *some* non-empty open subset of the manifold. Outside this set the operator can be anything and will not be modified. The precise statement is as follows

**Theorem 4.1.** *Let  $M$  be a closed Riemannian manifold of dimension  $n \equiv 3 \pmod{4}$ . Let  $D$  be an elliptic self-adjoint differential operator over  $M$  of order 1. Let  $U \subset M$  be a non-empty open subset. Let the restriction of  $D$  to  $U$  be a twisted Dirac operator.*

*Then one can deform the Riemannian metric in  $U$  such that the resulting operator  $\tilde{D}$  has non-trivial kernel.*

The proof relies firstly on the computation of the spectrum of the classical Dirac operator for a certain 1-parameter family of metrics on odd-dimensional spheres, the so-called *Berger metrics*. This computation has been carried out in [5].

Secondly, we prove a gluing theorem for such operators on connected sums. Given two closed manifolds of dimension  $n \geq 3$  together with operators which look like the classical Dirac operator (or a multiple of it) in some non-empty open subset, then by removing balls in these subsets and gluing one can form the connected sum with a Riemannian metric such that the spectrum of the associated operator on the sum is close to the disjoint union of the spectra of the two original operators, at least in some bounded range.

**Theorem 2.1.** (*Gluing Theorem*)

Let  $M_1$  and  $M_2$  be  $n$ -dimensional closed Riemannian manifolds of dimension  $n \geq 3$ . Let  $U_i \subset M_i$  be open balls, let  $D_i$  be self-adjoint elliptic differential operators of first order over  $M_i$  of Dirac type over  $U_i$ . Let  $\Lambda > 0$  such that  $\pm\Lambda \notin \text{spec}(D_1) \cup \text{spec}(D_2)$ . Let  $\epsilon > 0$ .

Then there exists a Riemannian metric on  $X = M_1 \# M_2$  and a self-adjoint elliptic first order differential operator  $D$  over  $X$  such that  $X$  is a disjoint union  $X = X_1 \dot{\cup} X_2 \dot{\cup} X_3$  where

- (i)  $X_1$  is isometric to  $M_1 - U_1$  and  $D$  coincides with  $D_1$  over  $X_1$ ,
- (ii)  $X_2$  is isometric to  $M_2 - U_2$  and  $D$  coincides with  $D_2$  over  $X_2$ ,
- (iii)  $X_3$  is diffeomorphic to  $(0, 1) \times S^{n-1}$  and  $D$  is of Dirac type over  $X_3$ ,

and such that  $D$  is  $(\Lambda, \epsilon)$ -spectral close to the disjoint union  $D_1 \dot{\cup} D_2$  over  $M_1 \dot{\cup} M_2$ .

In [5] we gave a proof of this Gluing Theorem for the classical Dirac operator on odd dimensional spin manifolds,  $n \geq 3$ . The restriction to odd dimension had technical reasons. We worked with explicit solutions of the eigenspinor equation on Euclidean annuli which can be given by a power series if the dimension is odd. In even dimensions additional logarithmic terms appear. Although it is likely that one can carry over the proof of [5] we chose to avoid the use of explicit solutions in this paper and we work with a-priori estimates instead.

In Section 2 we first describe the class of operators under consideration and then we formulate the Gluing Theorem. The proof is carried out except for the a-priori estimates on the distribution of the  $L^2$ -norm of eigenspinors on Euclidean annuli. They are derived in Section 3. In Section 4 we apply the Gluing Theorem and prove existence of metrics with harmonic sections.

An excellent introduction to the Dirac operator of spin and  $\text{spin}^c$  manifolds is given in [13] or in [8].  $\text{Spin}^h$  manifolds are explained in [4, 15]. The variational characterization of eigenvalues is explained in [10], at least for the Laplace operator.

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## 2 The Gluing Theorem

The aim of this section is to formulate, and partially prove, a theorem which allows us to compare the spectrum of twisted Dirac operators on two closed manifolds  $M_1$  and  $M_2$  with the spectrum of the corresponding operator on the connected sum  $M_1 \# M_2$  equipped with a suitable metric.

To start, we describe the class of operators that we will consider. If  $M$  is a closed Riemannian spin manifold, then there is a natural self-adjoint elliptic first order differential operator  $D$ , the *Dirac operator*. It acts on complex spinor fields and has discrete spectrum. Given an additional complex vector bundle  $E$  over  $M$  with connection, one can also form the twisted Dirac operator  $D^E$  acting on spinors with coefficients in  $E$ . See [13] for details. Since most of our considerations will be local in nature we will be able to avoid global topological conditions on  $M$  like the spin condition. We make the following

**Definition.** Let  $M$  be a closed Riemannian manifold, let  $U \subset M$  be open. A (formally) self-adjoint elliptic first order differential operator  $D$  acting on sections of a complex vector bundle over  $M$  will be called *of Dirac type over  $U$* , if  $D$  coincides over  $U$  with a multiple of the Dirac operator acting on spinors, i.e.  $D$  coincides over  $U$  with the Dirac operator twisted by a trivial flat bundle.

If the manifold is spin, we can take the Dirac operator itself as an example, possibly somehow deformed outside  $U$ . Another example is given by the Dirac operator of a  $\text{spin}^c$  manifold provided the canonical line bundle is trivial and flat over  $U$ . To get a convenient formulation of the Gluing Theorem we introduce the following terminology.

**Definition.** Let  $D_1$  and  $D_2$  be two self-adjoint operators with discrete spectrum  $\text{spec}(D_i)$ , let  $\Lambda > 0$ , and  $\epsilon > 0$ . We say that  $D_1$  and  $D_2$  are  $(\Lambda, \epsilon)$ -*spectral close* iff

- (i)  $+\Lambda, -\Lambda \notin \text{spec}(D_1) \cup \text{spec}(D_2)$
- (ii)  $D_1$  and  $D_2$  have the same number of eigenvalues in the interval  $(-\Lambda, \Lambda)$  (counted with multiplicities). Write

$$\begin{aligned} \text{spec}(D_1) \cap (-\Lambda, \Lambda) &= \{\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k\} \\ \text{spec}(D_2) \cap (-\Lambda, \Lambda) &= \{\mu_1 \leq \mu_2 \leq \dots \leq \mu_k\} \end{aligned}$$

- (iii)  $|\mu_j - \lambda_j| < \epsilon$  for  $j = 1, \dots, k$ .

Now we can formulate the main result of this section.

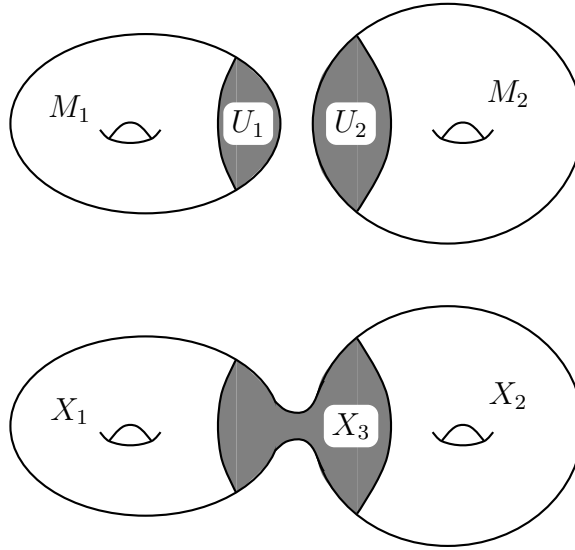
**Theorem 2.1.** (*Gluing Theorem*)

Let  $M_1$  and  $M_2$  be  $n$ -dimensional closed Riemannian manifolds of dimension  $n \geq 3$ . Let  $U_i \subset M_i$  be open balls, let  $D_i$  be self-adjoint elliptic differential operators of first order over  $M_i$  of Dirac type over  $U_i$ . Let  $\Lambda > 0$  such that  $\pm\Lambda \notin \text{spec}(D_1) \cup \text{spec}(D_2)$ . Let  $\epsilon > 0$ .

Then there exists a Riemannian metric on  $X = M_1 \# M_2$  and a self-adjoint elliptic first order differential operator  $D$  over  $X$  such that  $X$  is a disjoint union  $X = X_1 \dot{\cup} X_2 \dot{\cup} X_3$  where

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- (iii)  $X_3$  is diffeomorphic to  $(0, 1) \times S^{n-1}$  and  $D$  is of Dirac type over  $X_3$ ,

and such that  $D$  is  $(\Lambda, \epsilon)$ -spectral close to the disjoint union  $D_1 \dot{\cup} D_2$  over  $M_1 \dot{\cup} M_2$ .



**Fig. 1**

The operators that we have in mind for applications are locally twisted Dirac operators which are not necessarily globally twisted Dirac operators because the

manifold need not be spin. Examples are the Dirac operators for  $\text{spin}^c$  or  $\text{spin}^h$  manifolds.

It is an easy exercise to check that the Gluing Theorem does not hold in dimension 1, e.g. for the Dirac operator  $D = i \cdot \frac{d}{dt}$  on  $S_L^1 = \mathbb{R}/L \cdot \mathbb{Z}$ ,  $L > 0$ .

The rest of this and the next section are devoted to the proof of the Gluing Theorem. To start, we note that [5, Prop. 7.1] and its proof show the following:

There exists a neighborhood of the Riemannian metric  $g_i$  on  $M_i$  in the  $C^1$ -topology such that for every metric  $g'_i$  in this neighborhood which coincides with  $g_i$  outside  $U_i$  the corresponding operator  $D'_i$  (which coincides with  $D_i$  outside  $U_i$  and is of Dirac type over  $U_i$  for the metric  $g'_i$ ) and  $D_i$  are  $(\Lambda, \frac{\epsilon}{2})$ -spectral close, say. The Taylor expansion of a Riemannian metric in exponential coordinates shows that one can deform  $g_i$  in  $U_i$  such that  $U_i$  contains a small  $n$ -ball with Euclidean metric and this deformation can be made arbitrarily small in the  $C^1$ -topology. Hence we may assume w.l.o.g. that  $U_1$  and  $U_2$  contain small Euclidean  $n$ -balls of radius  $R > 0$ . Denote the centers of these balls by  $p_i$ .

Let  $0 < t_2 < \min\{R, 2^{-4}\}$ . Put  $t_{-2} := 2^{-9} \cdot t_2^{16}$ . Choose a smooth function  $\rho : \mathbb{R} \rightarrow \mathbb{R}$  with

- a)  $\rho(t) = |t|$  for  $|t| \geq t_{-2}$
- b)  $0 < \rho(t) \leq t_{-2}$  for  $|t| \leq t_{-2}$
- c)  $|\dot{\rho}(t)| \leq 1$  for all  $t$ .

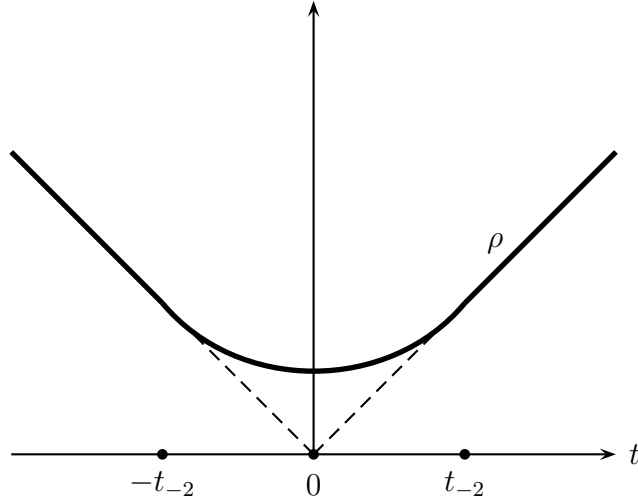


Fig. 2

We construct a metric on  $M_1 \# M_2$  as follows. On  $M_i - B(p_i, t_{-2})$  take the metric of  $M_i$ . Recall that this metric is Euclidean on  $B(p_i, R)$ . Remove the two balls  $B(p_1, t_{-2}) \subset M_1$  and  $B(p_2, t_{-2}) \subset M_2$  and replace them by the cylinder

$$[-t_{-2}, t_{-2}] \times S^{n-1}$$

with the warped product metric

$$ds^2 = dt^2 + \rho(t)^2 d\sigma^2$$

where  $d\sigma^2$  is the standard metric on  $S^{n-1}$  of constant sectional curvature 1. This yields a smooth Riemannian metric  $g_{t_2}$  on  $M_1 \# M_2$ .

Let  $D_{t_2}$  be the corresponding operator over  $X = M_1 \# M_2$  of Dirac type over  $[-t_{+2}, t_{+2}] \times S^n$  with respect to the metric  $g_{t_2}$ . For any operator  $D$ ,  $a \leq b$ , denote by  $E_{[a,b]}(D)$  the sum of all eigenspaces for the eigenvalues  $\lambda \in [a, b]$ . The Gluing Theorem follows readily from the following

CLAIM: Given  $\epsilon > 0$  and  $\Lambda > 0$  there exists  $\delta = \delta(\epsilon, \Lambda, k) > 0$  where  $k$  is the total number of eigenvalues of  $D_1$  and  $D_2$  in the interval  $(-\Lambda, \Lambda)$  such that for  $\lambda \in (-\Lambda + 2\epsilon, \Lambda - 2\epsilon)$  and  $0 < t_2 < \min\{\delta, R\}$  we have

$$\begin{aligned} \dim E_{\{\lambda\}}(D_1) + \dim E_{\{\lambda\}}(D_2) &\leq \dim E_{[\lambda-\epsilon, \lambda+\epsilon]}(D_{t_2}) \\ &\leq \dim E_{[\lambda-2\epsilon, \lambda+2\epsilon]}(D_1) + \dim E_{[\lambda-2\epsilon, \lambda+2\epsilon]}(D_2). \end{aligned}$$

In fact, it will turn out that  $\delta := \min \left\{ \frac{1}{100\Lambda^2}, 2^{-4}, \frac{1}{2\Lambda}, \frac{1}{2(k+1)}, 2^{-17} \cdot \frac{\epsilon^2}{(k+1)^2} \right\}$  does the job.

Why does the Claim imply the Gluing Theorem ? Let  $\Lambda > 0$  and  $\epsilon > 0$ . To see that the Gluing Theorem holds for the metric  $g_{t_2}$  if  $0 < t_2 < \min\{\delta, R\}$  let us assume w.l.o.g. that  $\epsilon$  is so small that any two distinct eigenvalues of  $D_1 \oplus D_2$  in  $(-\Lambda, \Lambda)$  have distance at least  $4\epsilon$  from one another and that they have distance at least  $2\epsilon$  from  $-\Lambda$  and from  $\Lambda$ .

If  $\lambda \in (-\Lambda, \Lambda)$  is an eigenvalue of  $D_1 \oplus D_2$ , then the Claim yields

$$\dim E_{\{\lambda\}}(D_1) + \dim E_{\{\lambda\}}(D_2) = \dim E_{[\lambda-\epsilon, \lambda+\epsilon]}(D_{t_2})$$

Thus for any eigenvalue of  $D_1 \oplus D_2$  of multiplicity  $m$  there are exactly  $m$  eigenvalues of  $D_{t_2}$  at distance at most  $\epsilon$ .

Are there further eigenvalues of  $D_{t_2}$  in  $(-\Lambda, \Lambda)$  ? Since such an eigenvalue  $\mu$  would have to have distance more than  $\epsilon$  from all eigenvalues of  $D_1$  and  $D_2$  we

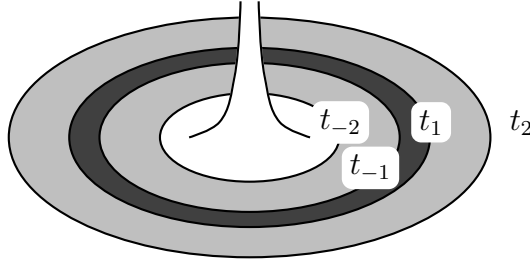


could find  $\lambda \in (-\Lambda, \Lambda)$  such that  $[\lambda - \epsilon, \lambda + \epsilon]$  contains  $\mu$  but  $[\lambda - 2\epsilon, \lambda + 2\epsilon]$  does not contain any eigenvalue of  $D_1$  or  $D_2$ . Then again by the Claim

$$\begin{aligned} 1 &\leq \dim E_{[\lambda - \epsilon, \lambda + \epsilon]}(D_{t_2}) \\ &\leq \dim E_{[\lambda - 2\epsilon, \lambda + 2\epsilon]}(D_1) + \dim E_{[\lambda - 2\epsilon, \lambda + 2\epsilon]}(D_2) \\ &= 0, \end{aligned}$$

a contradiction. This proves the Gluing Theorem.

To prove the Claim we set  $t_1 := \frac{1}{2} \cdot t_2^4, t_{-1} := \frac{1}{2}t_1$ .



**Fig. 3**

We choose smooth cut-off functions  $\chi_i : M_i \rightarrow \mathbb{R}$  such that

- a)  $0 \leq \chi_i \leq 1$
- b)  $\chi_i \equiv 1$  on  $M_i - B(p_i, t_1)$
- c)  $\chi_i \equiv 0$  on  $B(p_i, t_{-1})$
- d)  $|\nabla \chi_i| \leq \frac{2}{t_1 - t_{-1}} = \frac{4}{t_1}$ .

### **Proof of inequality**

$$\dim E_{\{\lambda\}}(D_1) + \dim E_{\{\lambda\}}(D_2) \leq \dim E_{[\lambda - \epsilon, \lambda + \epsilon]}(D_{t_2}).$$

The idea of proof of this inequality is as follows. Pick an eigenspinor  $\sigma$  for the eigenvalue  $\lambda$  on  $M_1$ , say. We multiply  $\sigma$  by the cut-off function  $\chi_1$  to obtain a

spinor  $\tilde{\sigma}$  on  $X$ . Of course,  $\tilde{\sigma}$  is no longer an eigenspinor. We use  $\tilde{\sigma}$  as a test function and plug it into the Rayleigh quotient for  $D_{t_2} - \lambda$  and have to show that this Rayleigh quotient is bounded by  $\epsilon$ . It then follows that  $D_{t_2}$  has an eigenvalue in the  $\epsilon$ -neighborhood of  $\lambda$ .

Multiplying by  $\chi_1$  makes the  $L^2$ -norm smaller, thus the denominator of the Rayleigh quotient becomes smaller. We have to show that we do not lose too much  $L^2$ -norm. Inequality (2.2) says that we lose at most half of the  $L^2$ -norm. This is not a very sharp estimate but sufficient for our purposes.

More seriously, in the numerator of the Rayleigh quotient there appears a very large error term involving the gradient of the cut-off function. But the support of the cut-off function is contained in the annulus  $Z_{t_{-1}, t_1}^n$  (dark grey in Figure 3). We show that the  $L^2$ -norm of  $\sigma$  in this annulus is so small compared to its whole  $L^2$ -norm that it overcompensates for the error term, see inequality (2.3).

All estimates in this section use the fact that the  $L^2$ -norm of eigenspinors on a Euclidean annulus is not arbitrarily distributed but tends to cumulate near the boundary of the annulus. In Figure 3 this means that the dark grey inner annulus carries very little  $L^2$ -norm compared to the light grey region. These “a-priori estimates” are proved in Section 3, see Proposition 3.2 and corollaries.

To prove the inequality we define linear maps  $E_{\{\lambda\}}(D_i) \rightarrow L^2(\Sigma X)$  by  $\sigma \rightarrow \tilde{\sigma} := \chi_i \cdot \sigma$ . We regard  $\tilde{\sigma}$  as a spinor over  $X$  by extending it by 0 in the obvious manner. By the unique-continuation property for Dirac operators [1, 9] these maps are 1 – 1. Since  $\chi_1$  and  $\chi_2$  have disjoint supports in  $X$ , the images  $\tilde{E}_{\{\lambda\}}(D_1)$  and  $\tilde{E}_{\{\lambda\}}(D_2)$  of  $E_{\{\lambda\}}(D_1)$  and  $E_{\{\lambda\}}(D_2)$  in  $L^2(\Sigma X)$  are  $L^2$ -orthogonal. In particular,

$$\dim(\tilde{E}_{\{\lambda\}}(D_1) + \tilde{E}_{\{\lambda\}}(D_2)) = \dim E_{\{\lambda\}}(D_1) + \dim E_{\{\lambda\}}(D_2).$$

We will show that the Rayleigh quotient of  $(D_{t_2} - \lambda)^2$  is bounded by  $\epsilon^2$  on  $\tilde{E}_{\{\lambda\}}(D_1) \oplus \tilde{E}_{\{\lambda\}}(D_2)$ , provided  $t_2 < \delta$ . This obviously implies the inequality in question. Since  $\tilde{E}_{\{\lambda\}}(D_1)$  and  $\tilde{E}_{\{\lambda\}}(D_2)$  are orthogonal we may look at both spaces separately.

Let  $\sigma \in E_{\{\lambda\}}(D_1)$ ,  $\tilde{\sigma} = \chi_1 \cdot \sigma \in \tilde{E}_{\{\lambda\}}(D_1)$ . Then

$$\begin{aligned} \frac{\|(D_{t_2} - \lambda)\tilde{\sigma}\|_{L^2(X)}^2}{\|\tilde{\sigma}\|_{L^2(X)}^2} &= \frac{\|\nabla \chi_1 \cdot \sigma + \chi_1 \cdot D_1 \sigma - \lambda \cdot \chi_1 \cdot \sigma\|_{L^2(X)}^2}{\|\tilde{\sigma}\|_{L^2(X)}^2} \\ &= \frac{\|\nabla \chi_1 \cdot \sigma\|_{L^2(X)}^2}{\|\tilde{\sigma}\|_{L^2(X)}^2}. \end{aligned} \tag{2.1}$$

By Corollary 2 to Proposition 3.2 (see next section) we know

$$\begin{aligned} \|\sigma\|_{L^2(Z_{0, t_1}^n)}^2 &\leq 2^9 \cdot t_1^2 \cdot t_2 \cdot \|\sigma\|_{L^2(Z_{0, t_2}^n)}^2 \\ &\leq 2^9 \cdot 2^{-12} \cdot \|\sigma\|_{L^2(M_1)}^2 \end{aligned} \quad \text{since } t_1 \leq t_2 \leq \delta \leq 2^{-4}$$

$$\leq \frac{1}{2} \cdot \|\sigma\|_{L^2(M_1)}^2.$$

Hence

$$\|\tilde{\sigma}\|_{L^2(X)}^2 \geq \|\sigma\|_{L^2(M_1)}^2 - \|\sigma\|_{L^2(Z_{0,t_1}^n)}^2 \geq \frac{1}{2} \|\sigma\|_{L^2(M_1)}^2. \quad (2.2)$$

Since  $t_2 < \delta \leq \frac{1}{100\Lambda^2}$  we can apply Corollary 1 to Proposition 3.2 and obtain, using property d) of  $\chi_1$

$$\begin{aligned} \|\nabla \chi_1 \cdot \sigma\|_{L^2(X)}^2 &= \|\nabla \chi_1 \cdot \sigma\|_{L^2(Z_{t_{-1},t_1}^n)}^2 \\ &\leq \|\nabla \chi_1\|_{L^\infty}^2 \cdot \|\sigma\|_{L^2(Z_{t_{-1},t_1}^n)}^2 \\ &\leq \frac{16}{t_1^2} \cdot 2^7 \cdot t_1^2 \cdot t_2 \cdot \|\sigma\|_{L^2(Z_{t_{-2},t_2}^n)}^2 \\ &\leq 2^{11} \cdot t_2 \cdot \|\sigma\|_{L^2(M_1)}^2. \end{aligned} \quad (2.3)$$

By (2.1), (2.2), and (2.3) we can estimate the Rayleigh quotient

$$\frac{\|(D_{t_2} - \lambda)\tilde{\sigma}\|_{L^2(X)}^2}{\|\tilde{\sigma}\|_{L^2(X)}^2} = \frac{\|\nabla \chi_1 \cdot \sigma\|_{L^2(X)}^2}{\|\tilde{\sigma}\|_{L^2(X)}^2} \leq 2^{12} \cdot t_2 \leq \epsilon^2$$

since  $t_2 < \delta < 2^{-12} \cdot \epsilon^2$ .

### Proof of inequality

$$\dim E_{[\lambda-\epsilon, \lambda+\epsilon]}(D_{t_2}) \leq \dim E_{[\lambda-2\epsilon, \lambda+2\epsilon]}(D_1) + \dim E_{[\lambda-2\epsilon, \lambda+2\epsilon]}(D_2).$$

This inequality is slightly more difficult to prove than the previous one. One additional difficulty comes from the fact that this time we do not start with a single eigenspinor but with a linear combination of eigenspinors for different eigenvalues. This is where the dependence of  $\delta$  on the total number  $k$  of eigenvalues of  $D_1$  and  $D_2$  in the interval  $(-\Lambda, \Lambda)$  comes into the game.

Secondly, since we start with eigenspinors on  $X$  we also have to control their  $L^2$ -norm on the connecting neck between  $M_1$  and  $M_2$ . Hence we need estimates for eigenspinors on cylindrical manifolds with certain warped product metrics. This is provided by Proposition 3.3. The rest of the proof is analogous to that of the previous inequality.

For  $-t_2 \leq \mu < \nu \leq t_2$  denote the connecting cylinder  $[\mu, \nu] \times S^{n-1} \subset X$  with metric  $ds^2 = dt^2 + \rho^2 d\sigma^2$  by  $Z_{\mu, \nu}$ .

We assume  $\Lambda \cdot t_2 \leq \Lambda \cdot \delta \leq \frac{1}{2}$  so that for  $|\lambda| \leq \Lambda$ :

$$|\lambda| \cdot |\rho| + \frac{1}{2} |\dot{\rho}| \leq \Lambda \cdot t_2 + \frac{1}{2} \cdot 1 \leq 1.$$

Hence we can apply Proposition 3.3 and obtain for any eigenspinor  $\sigma$  on  $Z_{-t_2, t_2}$  for the eigenvalue  $\lambda$

$$\begin{aligned}\|\sigma\|_{L^2(Z_{-t_1, t_1})}^2 &\leq \frac{t_1 - (-t_1)}{2(t_2 - t_1)} \cdot \|\sigma\|_{L^2(Z_{-t_2, t_2})}^2 \\ &= \frac{t_2^4}{2(1 - \frac{1}{2}t_2^3)t_2} \cdot \|\sigma\|_{L^2(Z_{-t_2, t_2})}^2 \\ &\leq t_2^3 \cdot \|\sigma\|_{L^2(Z_{-t_2, t_2})}^2.\end{aligned}$$

If  $\sigma = \sum \sigma_j$  is a sum of at most  $k + 1$   $D_{t_2}$ -eigenspinors on  $X$ , pairwise  $L^2(X)$ -orthogonal, then we conclude

$$\begin{aligned}\|\sigma\|_{L^2(Z_{-t_1, t_1})} &\leq \sum_j \|\sigma_j\|_{L^2(Z_{-t_1, t_1})} \\ &\leq t_2^{3/2} \cdot \sum_j \|\sigma_j\|_{L^2(Z_{-t_2, t_2})} \\ &\leq t_2^{3/2} \cdot \sum_j \|\sigma_j\|_{L^2(X)} \\ &\leq t_2 \cdot (k + 1) \cdot \|\sigma\|_{L^2(X)}.\end{aligned}$$

Since we assume  $t_2 < \delta \leq \frac{1}{2(k+1)}$ , we conclude

$$\|\sigma\|_{L^2(Z_{-t_1, t_1})} \leq \frac{1}{2} \|\sigma\|_{L^2(X)}. \quad (2.4)$$

Put  $l := \dim E_{[\lambda-2\epsilon, \lambda+2\epsilon]}(D_1) + \dim E_{[\lambda-2\epsilon, \lambda+2\epsilon]}(D_2)$ . We will derive a contradiction from the assumption

$$\dim E_{[\lambda-\epsilon, \lambda+\epsilon]}(D_{t_2}) \geq l + 1.$$

Let  $V$  be spanned by  $l + 1$  linearly independent  $D_{t_2}$ -eigenspinors for eigenvalues in  $[\lambda - \epsilon, \lambda + \epsilon]$ . Note that  $l + 1 \leq k + 1$ . The linear map

$$\begin{aligned}V &\rightarrow L^2(\Sigma M_1) \oplus L^2(\Sigma M_2) \\ \sigma &\rightarrow (\chi_1 \cdot \sigma, \chi_2 \cdot \sigma)\end{aligned}$$

is again 1-1 by the unique-continuation property. We will show that the Rayleigh quotient of  $(D_1 - \lambda)^2 \oplus (D_2 - \lambda)^2$  is bounded by  $(2\epsilon)^2$  on the image of  $V$ . This would imply that  $D_1 \oplus D_2$  has at least  $l + 1$  eigenvalues in  $[\lambda - 2\epsilon, \lambda + 2\epsilon]$ , a contradiction.

$$\begin{aligned}\|(\chi_1 \cdot \sigma, \chi_2 \cdot \sigma)\|_{L^2(M_1) \oplus L^2(M_2)}^2 &= \|\chi_1 \cdot \sigma\|_{L^2(M_1)}^2 + \|\chi_2 \cdot \sigma\|_{L^2(M_2)}^2 \\ &\geq \|\sigma\|_{L^2(X)}^2 - \|\sigma\|_{L^2(Z_{-t_1, t_1})}^2 \\ &\geq \frac{1}{2} \|\sigma\|_{L^2(X)}^2\end{aligned} \quad (2.5)$$

by (2.4). This implies for the Rayleigh quotient

$$\begin{aligned}
& \frac{\|(D_1 - \lambda) \oplus (D_2 - \lambda)(\chi_1 \sigma, \chi_2 \sigma)\|_{L^2(M_1) \oplus L^2(M_2)}^2}{\|(\chi_1 \sigma, \chi_2 \sigma)\|_{L^2(M_1) \oplus L^2(M_2)}^2} = \\
& = \frac{\|(D_1 - \lambda)(\chi_1 \sigma)\|_{L^2(M_1)}^2 + \|(D_2 - \lambda)(\chi_2 \sigma)\|_{L^2(M_2)}^2}{\|(\chi_1 \sigma, \chi_2 \sigma)\|_{L^2(M_1) \oplus L^2(M_2)}^2} \\
& \leq 2 \cdot \frac{\|(D_1 - \lambda)(\chi_1 \sigma)\|_{L^2(M_1)}^2 + \|(D_2 - \lambda)(\chi_2 \sigma)\|_{L^2(M_2)}^2}{\|\sigma\|_{L^2(X)}^2} \quad (2.6)
\end{aligned}$$

By property d) of  $\chi_i$  and Corollary 1 to Proposition 3.2 we know

$$\begin{aligned}
\|\nabla \chi_i \cdot \sigma\|_{L^2(M_i)} &= \|\nabla \chi_i \cdot \sigma\|_{L^2(Z_{t_{-1}, t_1}^n)} \\
&\leq \frac{4}{t_1} \cdot \|\sigma\|_{L^2(Z_{t_{-1}, t_1}^n)} \\
&\leq \frac{4}{t_1} \cdot \sqrt{2^7 \cdot t_1^2 \cdot t_2} \cdot \sum_j \|\sigma_j\|_{L^2(Z_{t_{-2}, t_2}^n)} \\
&\leq \sqrt{2^{11} \cdot t_2} \cdot (k+1) \cdot \|\sigma\|_{L^2(X)} \\
&\leq \frac{\epsilon}{8} \cdot \|\sigma\|_{L^2(X)} \quad (2.7)
\end{aligned}$$

because  $t_2 < \delta \leq 2^{-17} \cdot \frac{\epsilon^2}{(k+1)^2}$ . Hence, by (2.7)

$$\begin{aligned}
\frac{\|(D_i - \lambda)(\chi_i \sigma)\|_{L^2(M_i)}}{\|\sigma\|_{L^2(X)}} &= \frac{\|\chi_i(D_i - \lambda)\sigma + \nabla \chi_i \sigma\|_{L^2(M_i)}}{\|\sigma\|_{L^2(X)}} \\
&\leq \frac{\|\chi_i(D_i - \lambda)\sigma\|_{L^2(M_i)} + \|\nabla \chi_i \sigma\|_{L^2(M_i)}}{\|\sigma\|_{L^2(X)}} \\
&\leq \frac{\|(D_{t_2} - \lambda)(\sigma)\|_{L^2(M_i - B(p_i, t_{-1}))}}{\|\sigma\|_{L^2(X)}} + \frac{\epsilon}{8} \quad (2.8)
\end{aligned}$$

Combining (2.6) and (2.8) we obtain the desired estimate for the Rayleigh quotient:

$$\begin{aligned}
& \frac{\|(D_1 - \lambda) \oplus (D_2 - \lambda)(\chi_1 \sigma, \chi_2 \sigma)\|_{L^2(M_1) \oplus L^2(M_2)}^2}{\|(\chi_1 \sigma, \chi_2 \sigma)\|_{L^2(M_1) \oplus L^2(M_2)}^2} \leq \\
& \leq 2 \cdot \left\{ \left[ \frac{\|(D_{t_2} - \lambda)(\sigma)\|_{L^2(M_1 - B(p_1, t_{-1}))}}{\|\sigma\|_{L^2(X)}} + \frac{\epsilon}{8} \right]^2 + \right. \\
& \quad \left. + \left[ \frac{\|(D_{t_2} - \lambda)(\sigma)\|_{L^2(M_2 - B(p_2, t_{-1}))}}{\|\sigma\|_{L^2(X)}} + \frac{\epsilon}{8} \right]^2 \right\}
\end{aligned}$$

$$\begin{aligned}
&= 2 \cdot \left\{ \frac{\|(D_{t_2} - \lambda)(\sigma)\|_{L^2(X-Z_{-t_{-1}, t_{-1}})}^2}{\|\sigma\|_{L^2(X)}^2} + 2 \cdot \left(\frac{\epsilon}{8}\right)^2 + \right. \\
&\quad \left. + 2 \cdot \frac{\epsilon}{8} \cdot \frac{\|(D_{t_2} - \lambda)(\sigma)\|_{L^2(M_1-B(p_1, t_{-1}))} + \|(D_{t_2} - \lambda)(\sigma)\|_{L^2(M_2-B(p_2, t_{-1}))}}{\|\sigma\|_{L^2(X)}} \right\} \\
&\leq 2 \cdot \left\{ \epsilon^2 + 2 \cdot \left(\frac{\epsilon}{8}\right)^2 + 2 \cdot \frac{\epsilon}{8} \cdot 2\epsilon \right\} \\
&< (2\epsilon)^2.
\end{aligned}$$

This finishes the proof of the Claim and of the Gluing Theorem.  $\square$

### 3 The Estimates

In this section we will derive the a-priori estimates on the distribution of the  $L^2$ -norm of eigenspinors on Euclidean annuli which have been used in the previous section to prove the Gluing Theorem. Roughly speaking, what we show is the fact that the  $L^2$ -norm of an eigenspinor on a Euclidean annulus cumulates in the region near the boundary. In Figure 3 this means that there is only very little  $L^2$ -norm on the dark grey inner annulus  $Z_{t_{-1}, t_1}^n$  compared to the  $L^2$ -norm on the whole annulus  $Z_{t_{-2}, t_2}^n$ , compare Corollary 1 to Proposition 3.2. To prove this we regard a Euclidean annulus as a warped product of an interval with a sphere and decompose the eigenspinor with respect to an eigenbasis on the sphere. Then the eigenspinor equation translates into ordinary differential equations on the coefficients.

To control the coefficients we prove an estimate which allows us to compare the unknown solution of a linear ordinary differential equation with certain “almost solutions”. In the proof of Proposition 3.2 we will deal with linear ordinary differential equations which we cannot explicitly solve. But we will be able to guess the right “almost solutions” and thus obtain information on the unknown solutions. Proposition 3.1 and its proof are similar to the well-known Gronwall lemma.

**Proposition 3.1.** *Let  $I \subset \mathbb{R}$  be an interval,  $t_0 \in I$ , let  $A : I \rightarrow \text{Mat}(n \times n, \mathbb{C})$  be a continuous mapping into the complex  $n \times n$ -matrices. Let  $u$  be a solution of*

$$\dot{u}(t) = A(t)u(t).$$

*Moreover, let  $v : I \rightarrow \mathbb{C}^n$  be a continuously differentiable function such that*

- (i)  $v(t_0) = u(t_0)$
- (ii)  $|\dot{v}(t) - A(t)v(t)| \leq \delta(t)$

where  $\delta : I \rightarrow [0, \infty)$  is a continuous function.

Then the following estimate holds

$$|u(t) - v(t)| \leq \left| \int_{t_0}^t \delta(s) \cdot e^{\|A\|_\infty \cdot |t-s|} ds \right|$$

where  $\|A\|_\infty = \sup_{t \in I} |A(t)|_{\text{Op}}$  and  $|A(t)|_{\text{Op}}$  is the operator norm.

**Proof.** It is sufficient to prove the claim for  $t \geq t_0$ .  
Put  $w(t) := u(t) - v(t)$ . Then

$$\dot{w}(t) = A(t) \cdot w(t) + \rho(t)$$

where  $\rho(t) = A(t) \cdot v(t) - \dot{v}(t)$ . Thus

$$\begin{aligned} |w(t)| &= \left| \int_{t_0}^t (A(s) \cdot w(s) + \rho(s)) ds \right| \\ &\leq \|A\|_\infty \cdot \int_{t_0}^t |w(s)| ds + \int_{t_0}^t \delta(s) ds. \end{aligned} \tag{3.1}$$

For  $\varphi(t) := \|A\|_\infty \cdot \int_{t_0}^t |w(s)| ds + \int_{t_0}^t \delta(s) ds$

we have

$$\varphi(t_0) = 0$$

and

$$\dot{\varphi}(t) = \|A\|_\infty |w(t)| + \delta(t). \tag{3.2}$$

Hence by (3.1) and (3.2)

$$\begin{aligned} 0 &\leq \|A\|_\infty \cdot (\varphi(t) - |w(t)|) \\ &= \|A\|_\infty \cdot \varphi(t) - \dot{\varphi}(t) + \delta(t). \end{aligned}$$

Multiplying by  $e^{-\|A\|_\infty \cdot (t-t_0)}$  we obtain

$$0 \leq -\frac{d}{dt} \left( e^{-\|A\|_\infty \cdot (t-t_0)} \cdot \varphi(t) \right) + e^{-\|A\|_\infty \cdot (t-t_0)} \cdot \delta(t).$$

Integration yields

$$e^{-\|A\|_\infty \cdot (t-t_0)} \cdot \varphi(t) \leq \int_{t_0}^t \left( e^{-\|A\|_\infty \cdot (s-t_0)} \cdot \delta(s) \right) ds$$

which implies again using (3.1)

$$\begin{aligned} |w(t)| &\leq \varphi(t) \\ &\leq \int_{t_0}^t e^{\|A\|_\infty(t-s)} \cdot \delta(s) ds. \end{aligned} \quad \square$$

For  $0 < a < b$  let  $Z_{a,b}^n := \{x \in \mathbb{R}^n \mid a \leq |x| \leq b\}$  be the corresponding *Euclidean annulus*. As a Riemannian manifold we can also describe  $Z_{a,b}^n$  as a warped product

$$Z_{a,b}^n = [a, b] \times S^{n-1}$$

with the metric

$$ds^2(t, y) = dt^2 + t^2 d\sigma^2(y)$$

where  $d\sigma^2$  is the standard metric on  $S^{n-1}$  of constant sectional curvature 1.

Restriction of the spinor bundle  $\Sigma\mathbb{R}^n$  of  $\mathbb{R}^n$  to  $S^{n-1}$  yields the spinor bundle of  $S^{n-1}$  if  $n$  is odd. If  $n$  is even one obtains the direct sum of two copies of the spinor bundle on  $S^{n-1}$ . If  $n$  is odd let  $\tilde{D}$  be the Dirac operator of  $S^{n-1}$ , if  $n$  is even let  $\tilde{D}$  be the sum of the Dirac operator of  $S^{n-1}$  and of its negative.

We choose an orthonormal basis of  $L^2(\Sigma\mathbb{R}^n|_{S^{n-1}})$  consisting of eigenspinors  $\sigma_j$  of  $\tilde{D}$ . Clifford multiplication by the normal vector field  $\frac{\partial}{\partial t}$  anticommutes with  $\tilde{D}$ . Thus if  $\sigma_j$  is an eigenspinor for the eigenvalue  $\mu_j$ , then  $\frac{\partial}{\partial t} \cdot \sigma_j$  is an eigenspinor for  $-\mu_j$ . Hence we may assume  $\sigma_{-j} = \frac{\partial}{\partial t} \cdot \sigma_j$ ,  $j \in \mathbb{N}$ ,  $0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \dots \rightarrow +\infty$ . Parallel translation along the  $t$ -lines yields smooth sections  $(t, y) \mapsto \sigma_j(t, y)$  on  $(0, \infty) \times S^{n-1} = \mathbb{R}^n - \{0\}$ ,  $j \in \mathbb{Z}^* = \mathbb{Z} - \{0\} = \mathbb{N} \cup -\mathbb{N}$ . If  $\sigma$  is a smooth spinor field on  $Z_{a,b}^n$ , then we can write

$$\sigma(t, y) = \sum_{j \in \mathbb{Z}^*} \beta_j(t) \cdot t^{-\frac{n-1}{2}} \cdot \sigma_j(t, y)$$

where  $\beta_j$  are smooth functions,  $\beta_j : [a, b] \rightarrow \mathbb{C}$ . The normalization factor  $t^{-\frac{n-1}{2}}$  is chosen such that the subsequent formulas become very simple. If  $D$  is the Dirac operator on  $Z_{a,b}^n$ , then the eigenspinor equation

$$D\sigma = \lambda \cdot \sigma$$

is equivalent to

$$\dot{B}_j(t) = \begin{pmatrix} \frac{\mu_j}{t} & -\lambda \\ \lambda & -\frac{\mu_j}{t} \end{pmatrix} \cdot B_j(t) \quad (3.3)$$



where  $B_j(t) = \begin{pmatrix} \beta_{-j}(t) \\ \beta_j(t) \end{pmatrix}$  for all  $j \in \mathbb{N}$ . The  $L^2$ -norm of  $\sigma$  is given by

$$\|\sigma\|_{L^2(Z_{a,b}^n)}^2 = \sum_{j=1}^{\infty} \int_a^b |B_j(t)|^2 dt.$$

See [5] for the details. Now we come to the main estimate of this section.

**Proposition 3.2.** *Let  $0 < t_{-2} < t_{-1} < t_1 < t_2 \leq 1$ , let  $\mu \geq 1$ , let  $\lambda \in \mathbb{R}$ . If  $|\lambda|t_2^{1/2} \leq \frac{1}{10}$ ,  $t_2 \geq 2t_1$ ,  $t_{-1} \geq 2t_{-2}$ , and  $t_1^6 \leq t_{-2}$ , then every solution  $B$  of*

$$\dot{B}(t) = \begin{pmatrix} \frac{\mu}{t} & -\lambda \\ \lambda & -\frac{\mu}{t} \end{pmatrix} \cdot B(t)$$

*satisfies*

$$\frac{\int_{t_{-1}}^{t_1} |B(t)|^2 dt}{\int_{t_{-2}}^{t_2} |B(t)|^2 dt} \leq 2^6 \cdot \max \left\{ 3 \cdot \left( \frac{t_1}{t_2} \right)^{2\mu+1}, \frac{t_1}{t_{-1}} \cdot \left( \frac{t_{-2}}{t_{-1}} \right)^{2\mu-1} \right\}$$

**Corollary 1.** *Let  $n \geq 3$ , let  $0 < t_2 < 2^{-4}$ , let  $\Lambda > 0$ . Put  $t_1 := \frac{1}{2} \cdot t_2^4$ ,  $t_{-1} := \frac{1}{2}t_1$ ,  $t_{-2} := \frac{1}{2}t_{-1}^4$ . Then every eigenspinor  $\sigma$  for the eigenvalue  $\lambda$  on  $Z_{t_{-2}, t_2}^n$  satisfies*

$$\frac{\|\sigma\|_{L^2(Z_{t_{-1}, t_1}^n)}^2}{\|\sigma\|_{L^2(Z_{t_{-2}, t_2}^n)}^2} \leq 2^7 \cdot t_1^2 \cdot t_2,$$

*provided  $|\lambda| \leq \Lambda$  and  $\Lambda \cdot t_2^{1/2} \leq \frac{1}{10}$ .*

**Proof of Corollary 1.** With our definition of  $t_1$ ,  $t_{-1}$ , and  $t_{-2}$  the assumptions on  $t_j$  in Proposition 3.2 are trivially satisfied. Moreover, it is well-known that the absolute value of all Dirac eigenvalues of  $S^{n-1}$  is at least 1 if  $n \geq 3$ , compare e.g. [6]. Hence

$$\begin{aligned} \|\sigma\|_{L^2(Z_{t_{-1}, t_1}^n)}^2 &= \sum_{j=1}^{\infty} \int_{t_{-1}}^{t_1} |B_j(t)|^2 dt \\ &\leq 2^6 \cdot \sum_{j=1}^{\infty} \max \left\{ 3 \cdot \left( \frac{1}{2}t_2^3 \right)^{2\mu_j+1}, 2 \cdot \left( \frac{1}{2}t_{-1}^3 \right)^{2\mu_j-1} \right\} \cdot \int_{t_{-2}}^{t_2} |B_j(t)|^2 dt \end{aligned}$$

$$\begin{aligned}
&\leq 2^6 \cdot \max \left\{ 3 \cdot 2^{-3} \cdot t_2^9, t_{-1}^3 \right\} \cdot \|\sigma\|_{L^2(Z_{t_{-2}, t_2}^n)}^2 \\
&\leq 2^7 \cdot t_1^2 \cdot t_2 \cdot \|\sigma\|_{L^2(Z_{t_{-2}, t_2}^n)}^2.
\end{aligned}
\quad \square$$

**Corollary 2.** *Let  $n \geq 3$ , let  $0 < t_2 < 2^{-4}$ , let  $t_1 := \frac{1}{2}t_2^4$ , let  $\Lambda > 0$ . Then every eigenspinor  $\sigma$  for the eigenvalue  $\lambda$  on the ball  $Z_{0, t_2}^n$  satisfies*

$$\frac{\|\sigma\|_{L^2(Z_{0, t_1}^n)}^2}{\|\sigma\|_{L^2(Z_{0, t_2}^n)}^2} \leq 2^9 \cdot t_1^2 \cdot t_2$$

if  $|\lambda| \leq \Lambda$  and  $\Lambda t_2^{1/2} \leq \frac{1}{10}$ .

**Proof of Corollary 2.** We apply Corollary 1 to  $(k \in \mathbb{N}_0)$

$$\begin{aligned}
t_{2,k} &:= 2^{-k/4} \cdot t_2, \\
t_{1,k} &:= 2^{-k} \cdot t_1, \\
t_{-1,k} &:= \frac{1}{2} t_{1,k} = 2^{-k-1} \cdot t_1, \\
t_{-2,k} &:= \frac{1}{2} t_{-1,k}^4 :
\end{aligned}$$

$$\begin{aligned}
\|\sigma\|_{L^2(Z_{t_{-1,k}, t_{1,k}}^n)}^2 &\leq 2^7 \cdot t_{1,k}^2 \cdot t_{2,k} \cdot \|\sigma\|_{L^2(Z_{t_{-2,k}, t_{2,k}}^n)}^2 \\
&\leq 2^7 \cdot 2^{-2k} \cdot t_1^2 \cdot t_2 \cdot \|\sigma\|_{L^2(Z_{0, t_2}^n)}^2.
\end{aligned}$$

Summation over  $k$  yields

$$\|\sigma\|_{L^2(Z_{0, t_1}^n)}^2 \leq 2^7 \cdot \frac{4}{3} \cdot t_1^2 \cdot t_2 \cdot \|\sigma\|_{L^2(Z_{0, t_2}^n)}^2
\quad \square$$

**Proof of Proposition 3.2.** The substitution  $t = e^\tau$ ,  $\tilde{B}(\tau) = B(e^\tau)$ , translates our differential equation (3.3) into

$$\tilde{B}'(\tau) = \begin{pmatrix} \mu & -\lambda e^\tau \\ \lambda e^\tau & -\mu \end{pmatrix} \cdot \tilde{B}(\tau). \quad (3.4)$$

Set  $t_0 := \sqrt{t_1 \cdot t_{-1}}$ ,  $t_k = e^{\tau_k}$ ,  $k = -2, -1, 0, 1, 2$ . Write

$$B(t_0) = \tilde{B}(\tau_0) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

We will compare the solution  $\tilde{B}$  of (3.4) to the “almost solution”

$$\begin{aligned}\tilde{v}(\tau) &:= \begin{pmatrix} w_1 \cdot e^{\mu(\tau-\tau_0)} \\ w_2 \cdot e^{-\mu(\tau-\tau_0)} \end{pmatrix}, \\ v(t) &:= \begin{pmatrix} w_1 \cdot \left(\frac{t}{t_0}\right)^\mu \\ w_2 \cdot \left(\frac{t}{t_0}\right)^{-\mu} \end{pmatrix}.\end{aligned}$$

Define

$$A(\tau) := \begin{pmatrix} \mu & -\lambda e^\tau \\ \lambda e^\tau & -\mu \end{pmatrix}.$$

Then on  $[\tau_{-2}, \tau_2]$  we have  $\|A\|_\infty \leq \mu + \frac{1}{10}$  because  $|\lambda e^\tau| \leq |\lambda t_2| \leq |\lambda t_2^{1/2}| \leq \frac{1}{10}$ . We compute

$$\begin{aligned}|\tilde{v}' - A(\tau) \cdot \tilde{v}(\tau)|^2 &= \left| \lambda \cdot e^\tau \cdot \begin{pmatrix} w_2 \cdot e^{-\mu(\tau-\tau_0)} \\ -w_1 \cdot e^{\mu(\tau-\tau_0)} \end{pmatrix} \right|^2 \\ &= \lambda^2 \cdot e^{2\tau} \cdot (|w_1|^2 e^{2\mu(\tau-\tau_0)} + |w_2|^2 e^{-2\mu(\tau-\tau_0)}).\end{aligned}$$

Now we estimate the difference between the solution  $\tilde{B}$  and the “almost solution”  $\tilde{v}$ .

We start with the case that  $\tau \in [\tau_{-2}, \tau_0]$ .

We set  $\delta(\tau) := |\lambda| \cdot e^\tau \cdot |B(t_0)| \cdot e^{-\mu(\tau-\tau_0)}$ . We have

$$|\tilde{v}'(\tau) - A(\tau) \cdot \tilde{v}(\tau)| \leq \delta(\tau).$$

Proposition 3.1. yields

$$\begin{aligned}|\tilde{B}(\tau) - \tilde{v}(\tau)| &\leq \int_\tau^{\tau_0} \delta(\sigma) \cdot e^{\|A\|_\infty \cdot (\sigma-\tau)} d\sigma \\ &\leq |\lambda| \cdot |B(t_0)| \cdot \int_\tau^{\tau_0} e^\sigma \cdot e^{-\mu(\sigma-\tau_0)} \cdot e^{(\mu+\frac{1}{10})(\sigma-\tau)} d\sigma \\ &= |\lambda| \cdot |B(t_0)| \cdot e^{-\mu(\tau-\tau_0)} \cdot e^{-\frac{\tau}{10}} \cdot \int_\tau^{\tau_0} e^{\frac{11}{10}\sigma} d\sigma \\ &\leq |\lambda| \cdot |B(t_0)| \cdot e^{-\mu(\tau-\tau_0)} \cdot e^{\tau_0} \cdot e^{\frac{1}{10}(\tau_0-\tau)}.\end{aligned}$$

Thus

$$|B(t) - v(t)| \leq |\lambda| \cdot t_0 \cdot |B(t_0)| \cdot \left(\frac{t_0}{t_{-2}}\right)^{\frac{1}{10}} \cdot \left(\frac{t}{t_0}\right)^{-\mu}$$

$$= |\lambda| \cdot t_0^{1/2} \cdot |B(t_0)| \cdot \left(\frac{t_0^6}{t_{-2}}\right)^{\frac{1}{10}} \cdot \left(\frac{t}{t_0}\right)^{-\mu} \quad (3.5)$$

$$\leq |B(t_0)| \cdot \left(\frac{t}{t_0}\right)^{-\mu} \quad (3.6)$$

because  $t_0^6 \leq t_1^6 \leq t_{-2}$  and  $|\lambda| \cdot t_0^{1/2} \leq |\lambda| \cdot t_2^{1/2} < 1$ .

In case  $\tau \in [\tau_0, \tau_2]$  similar reasoning yields

$$|B(t) - v(t)| \leq \frac{10}{9} \cdot |\lambda| \cdot t \cdot |B(t_0)| \cdot \left(\frac{t}{t_0}\right)^\mu \quad (3.7)$$

$$\leq |B(t_0)| \cdot \left(\frac{t}{t_0}\right)^\mu. \quad (3.8)$$

Now that we have some control on how much  $v$  can differ from the solution  $B$  we can look at the growth of  $B$ . An upper bound is now easily obtained.

If  $t \in [t_0, t_2]$  we get by (3.8):

$$|B(t)| \leq |v(t)| + |B(t) - v(t)| \leq 2 \cdot |B(t_0)| \cdot \left(\frac{t}{t_0}\right)^\mu$$

and similarly for  $t \in [t_{-2}, t_0]$ :

$$|B(t)| \leq 2 \cdot |B(t_0)| \cdot \left(\frac{t}{t_0}\right)^{-\mu}.$$

From this we get

$$\begin{aligned} \int_{t_{-1}}^{t_1} |B(t)|^2 dt &= \int_{t_{-1}}^{t_0} |B(t)|^2 dt + \int_{t_0}^{t_1} |B(t)|^2 dt \\ &\leq 4 \cdot |B(t_0)|^2 \cdot \left\{ \frac{1}{2\mu - 1} \cdot \left[ \left(\frac{t_{-1}}{t_0}\right)^{-2\mu} \cdot t_{-1} - t_0 \right] \right. \\ &\quad \left. + \frac{1}{2\mu + 1} \left[ \left(\frac{t_1}{t_0}\right)^{2\mu} \cdot t_1 - t_0 \right] \right\} \\ &\leq \frac{8}{2\mu - 1} \cdot |B(t_0)|^2 \cdot \left(\frac{t_1}{t_{-1}}\right)^\mu \cdot t_1. \end{aligned} \quad (3.9)$$

by the definition of  $t_0$ .

To get a lower bound on  $|B(t)|$  we have to distinguish two cases.

**Case 1.**  $|w_1| \geq |w_2|$ .

We know  $|v(t)|^2 \geq |w_1|^2 \cdot \left(\frac{t}{t_0}\right)^{2\mu} \geq \frac{1}{2}|B(t_0)|^2 \cdot \left(\frac{t}{t_0}\right)^{2\mu}$ . For  $t \in [t_0, t_2]$  we get

$$\begin{aligned} |B(t)| &\geq |v(t)| - |B(t) - v(t)| \\ &\geq |B(t_0)| \cdot \left(\frac{t}{t_0}\right)^\mu \cdot \left\{ \frac{1}{\sqrt{2}} - \frac{10}{9} \cdot |\lambda| \cdot t \right\} \\ &\geq \frac{1}{2} \cdot |B(t_0)| \cdot \left(\frac{t}{t_0}\right)^\mu. \end{aligned} \quad \text{by (3.7)}$$

Hence by (3.6)

$$\begin{aligned} \int_{t_{-2}}^{t_2} |B(t)|^2 dt &\geq \int_{t_0}^{t_2} |B(t)|^2 dt \\ &\geq \frac{1}{4} \cdot |B(t_0)|^2 \cdot \frac{1}{2\mu+1} \cdot \left\{ \left(\frac{t_2}{t_0}\right)^{2\mu} \cdot t_2 - t_0 \right\} \\ &\geq \frac{1}{8} \cdot |B(t_0)|^2 \cdot \frac{1}{2\mu+1} \cdot \left(\frac{t_2}{t_0}\right)^{2\mu} \cdot t_2 \end{aligned} \quad (3.10)$$

because  $t_2 \geq 2 \cdot t_0$ . From (3.9), (3.10), and  $t_0^2 = t_1 \cdot t_{-1}$  we deduce

$$\begin{aligned} \frac{\int_{t_{-1}}^{t_1} |B(t)|^2 dt}{\int_{t_{-2}}^{t_2} |B(t)|^2 dt} &\leq \frac{\frac{8}{2\mu-1} \cdot |B(t_0)|^2 \cdot \left(\frac{t_1}{t_{-1}}\right)^\mu \cdot t_1}{\frac{1}{8} \cdot |B(t_0)|^2 \cdot \frac{1}{2\mu+1} \cdot \left(\frac{t_2}{t_0}\right)^{2\mu} \cdot t_2} \cdot \frac{t_1^\mu \cdot t_{-1}^\mu}{t_0^{2\mu}} \\ &= 2^6 \cdot \frac{2\mu+1}{2\mu-1} \cdot \left(\frac{t_1}{t_2}\right)^{2\mu+1} \\ &\leq 2^6 \cdot 3 \cdot \left(\frac{t_1}{t_2}\right)^{2\mu+1}. \end{aligned}$$

**Case 2.**  $|w_1| \leq |w_2|$ .

This time we have  $|v(t)|^2 \geq |w_2|^2 \cdot \left(\frac{t}{t_0}\right)^{-2\mu} \geq \frac{1}{2}|B(t_0)|^2 \cdot \left(\frac{t}{t_0}\right)^{-2\mu}$ . For  $t \in [t_{-2}, t_0]$  we get

$$\begin{aligned} |B(t)| &\geq |v(t)| - |B(t) - v(t)| \\ &\geq |B(t_0)| \cdot \left(\frac{t}{t_0}\right)^{-\mu} \cdot \left\{ \frac{1}{\sqrt{2}} - |\lambda| \cdot t_0^{1/2} \right\} \\ &\geq \frac{1}{2} \cdot |B(t_0)| \cdot \left(\frac{t}{t_0}\right)^{-\mu}. \end{aligned} \quad \text{by (3.5)}$$

Thus

$$\begin{aligned}
\int_{t_{-2}}^{t_2} |B(t)|^2 dt &\geq \int_{t_{-2}}^{t_0} |B(t)|^2 dt \\
&\geq \frac{1}{4} \cdot |B(t_0)|^2 \cdot \frac{1}{2\mu - 1} \cdot \left\{ \left( \frac{t_{-2}}{t_0} \right)^{-2\mu} \cdot t_{-2} - t_0 \right\} \\
&\geq \frac{|B(t_0)|^2}{8(2\mu - 1)} \cdot \left( \frac{t_0}{t_{-2}} \right)^{2\mu} \cdot t_{-2}
\end{aligned} \tag{3.11}$$

because  $t_0 \geq t_{-1} \geq 2t_{-2}$  and thus  $\left( \frac{t_0}{t_{-2}} \right)^{2\mu} - \left( \frac{t_0}{t_{-2}} \right) \geq \frac{1}{2} \left( \frac{t_0}{t_{-2}} \right)^{2\mu}$ .

Hence by (3.9) and (3.11)

$$\frac{\int_{t_{-1}}^{t_1} |B(t)|^2 dt}{\int_{t_{-2}}^{t_2} |B(t)|^2 dt} \leq \frac{\frac{8}{2\mu-1} \cdot |B(t_0)|^2 \cdot \left( \frac{t_1}{t_{-1}} \right)^\mu \cdot t_1}{\frac{|B(t_0)|^2}{8(2\mu-1)} \cdot \left( \frac{t_0}{t_{-2}} \right)^{2\mu} \cdot t_{-2}} = 2^6 \cdot \left( \frac{t_{-2}}{t_{-1}} \right)^{2\mu} \cdot \frac{t_1}{t_{-2}}.$$

This finishes the proof of Proposition 3.2.  $\square$

To conclude this section we cite the following proposition which has also been used in the proof of the Gluing Theorem.

**Proposition 3.3.** *Let  $n \geq 3$ . Let  $a, b, c, \lambda \in \mathbb{R}$  such that  $a < b$  and  $c > 0$ . Let  $\rho : [a - c, b + c] \rightarrow \mathbb{R}$  be a smooth positive function such that*

$$|\lambda| \cdot \|\rho\|_{L^\infty(a-c, b+c)} + \frac{1}{2} \cdot \|\dot{\rho}\|_{L^\infty(a-c, b+c)} \leq 1.$$

*Then for every Dirac eigenspinor  $\sigma$  for the eigenvalue  $\lambda$  on  $Z_{a-c, b+c} := [a - c, b + c] \times S^{n-1}$  with the warped product metric  $ds^2 = dt^2 + \rho(t)^2 d\sigma^2$  the following estimate holds*

$$\|\sigma\|_{L^2(Z_{a,b})}^2 \leq \frac{b-a}{2c} \left\{ \|\sigma\|_{L^2(Z_{b,b+c})}^2 + \|\sigma\|_{L^2(Z_{a-c,a})}^2 \right\}.$$

The proof can be found in [5, Prop. 5.1].

## 4 Harmonic Sections

In this last section we apply the Gluing Theorem to prove

**Theorem 4.1.** *Let  $M$  be a closed Riemannian manifold of dimension  $n \equiv 3 \pmod{4}$ . Let  $D$  be an elliptic self-adjoint differential operator over  $M$  of order 1. Let  $U \subset M$  be a non-empty open subset. Let the restriction of  $D$  to  $U$  be a twisted Dirac operator.*

*Then one can deform the Riemannian metric in  $U$  such that the resulting operator  $\tilde{D}$  has non-trivial kernel.*

Note that  $\tilde{D}$  coincides with  $D$  outside  $U$  and that the connection of the coefficient bundle over  $U$  is **not** modified. Theorem 4.1 applies for example to the Dirac operators of  $\text{spin}$ ,  $\text{spin}^c$ , or  $\text{spin}^h$  manifolds. By a suitable local deformation of the Riemannian metric while keeping the connection of the canonical line bundle fixed one can produce a non-trivial kernel for the Dirac operator of a  $\text{spin}^c$  manifold.

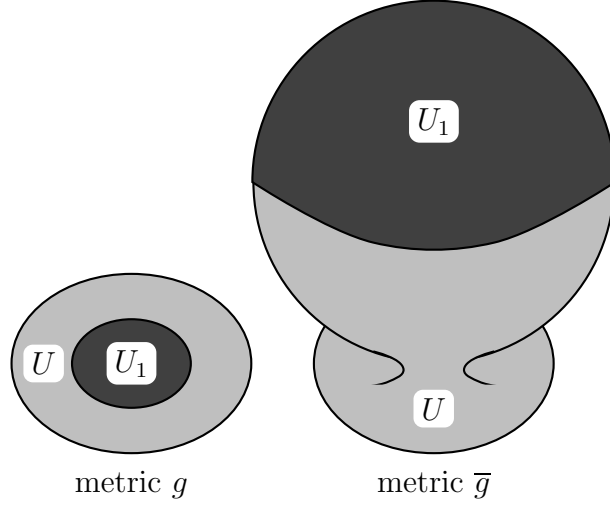
Theorem 4.1 does not apply to the Euler operator  $d + \delta$  acting on forms even though  $d + \delta$  is also a twisted Dirac operator (twisted by spinors). The point is that in this case a change of the Riemannian metric also changes the connection on the coefficient bundle.

From the discussion in the introduction we know that Theorem 4.1 is not true in dimension 2. The classical Dirac operator on  $S^2$  has no harmonic spinors no matter which Riemannian metric on  $S^2$  has been chosen.

**Proof of Theorem 4.1.** Over  $U$  our operator  $D$  is a twisted Dirac operator, i.e.  $D|_U = D^E$  where  $E$  is some bundle over  $U$  with connection  $\nabla^E$ . Choose a small  $n$ -ball  $U_1 \subset\subset U$ . By blowing up the metric  $g$  in a neighborhood of  $\overline{U}_1$  to some metric  $\overline{g}$  one can make

$$f = \text{id} : (U_1, \overline{g}) \rightarrow (U_1, g)$$

$\epsilon$ -contracting for an arbitrarily small prescribed  $\epsilon > 0$ .



**Fig. 4**

After trivializing the bundle  $E$  over  $U_1$  we can write

$$\nabla^E = \partial + \Gamma$$

where  $\partial$  is a flat coordinate derivative and  $\Gamma$  is an  $\text{End}(E)$ -valued 1-form given by the Christoffel symbols.

For the pull-back of a connection by a map  $f$  one has

$$f^*\nabla^E = \partial + \Gamma \circ df.$$

Since in our case  $f = \text{id}$  is  $\epsilon$ -contracting we see that with respect to the new metric  $\bar{g}$  we have

$$\|\Gamma\|_{L^\infty(U_1)} < \epsilon'$$

for small  $\epsilon' > 0$ . Choose a smaller  $n$ -ball  $U_2 \subset\subset U_1$  and a cut-off function  $\chi : M \rightarrow \mathbb{R}$  such that

- a)  $0 \leq \chi \leq 1$
- b)  $\chi \equiv 0$  on  $U_2$
- c)  $\chi \equiv 1$  on  $M - U_1$ .

Define a new connection  $\bar{\nabla}$  on  $E$  by

$$\bar{\nabla} := \partial + \chi \cdot \Gamma.$$



Then  $\overline{\nabla}$  is flat over  $U_2$  and  $\overline{\nabla} = \nabla^E$  over  $U - U_1$ . Denote by  $D_1$  the twisted Dirac operator for the original connection  $\nabla^E$  and the new metric  $\overline{g}$ , by  $D_2$  the twisted Dirac operator for  $\overline{\nabla}$  and  $\overline{g}$ . We extend  $D_1$  and  $D_2$  to  $M$  such that over  $M - U$  we have  $D_1 = D_2 = D$ . Then the difference  $D_1 - D_2$  is an operator of order 0 with support in  $U_1$ . Its  $L^2$ -operator norm can be estimated by

$$\|D_1 - D_2\|_{L^2, L^2} \leq n \cdot \|\overline{\nabla} - \nabla^E\|_{L^\infty} < n \cdot \epsilon' =: \epsilon''.$$

In particular,  $D_1$  and  $D_2$  are  $(\infty, \epsilon'')$ -spectral close and  $D_2$  is of Dirac type over  $U_2$ .

In [5, Section 3] it is shown that in dimension  $n \equiv 3 \pmod{4}$  there exists a one-parameter family  $g_T$  of Riemannian metrics on  $S^n$ ,  $T \in [a, b]$ , such that the following holds for the associated Dirac operator  $D_T$ :

- a) There is  $\lambda(T) \in \text{spec}(D_T)$  with  $\lambda(a) = -1$ ,  $\lambda(b) = +1$ .
- b)  $\lambda(T)$  depends smoothly (actually linearly) on  $T$ .
- c) The multiplicity of  $\lambda(T)$  is constant in  $T$  and can be chosen arbitrarily large.
- d)  $\lambda(T)$  is the only eigenvalue of  $D_T$  in the interval  $[-1, 1]$ .

We choose the multiplicity of  $\lambda(T)$  larger than the total number of eigenvalues of  $D_2$  in the interval  $[-1, 1]$ . Now we can apply the Gluing Theorem to  $D_2$  and  $D_T$ . We obtain a metric  $\overline{g}_T$  on  $M \# S^n = M$  such that the corresponding operator  $\overline{D}_T$  on  $M$  is  $(1, \epsilon'')$ -spectral close to  $D_2 \dot{\cup} D_T$ .

By construction of  $\overline{g}_T$  the identity mapping

$$\text{id} : (U_1, \overline{g}_T) \rightarrow (U_1, g)$$

is still  $\epsilon$ -contracting. Thus  $\tilde{D}_T$  and  $\overline{D}_T$  are  $(\infty, \epsilon'')$ -spectral close where  $\tilde{D}_T$  coincides with  $D$  outside  $U$  while over  $U$ ,  $\tilde{D}_T$  is the twisted Dirac operator for  $\nabla^E$  and  $\overline{g}_T$ . It follows that  $\tilde{D}_T$  is  $(1, 2\epsilon'')$ -spectral close to  $D_2 \dot{\cup} D_T$ .

Since the multiplicity of  $\lambda(T)$  was chosen larger than the total number of eigenvalues of  $D_2$  in  $[-1, 1]$ , there are more negative eigenvalues of  $D_2 \dot{\cup} D_T$  in  $[-1, 1]$  for  $T = a$  whereas there are more positive ones for  $T = b$ . If  $\epsilon''$  was chosen smaller than  $\frac{1}{2}$ , then this will remain true for the operator  $\tilde{D}_T$ . In particular, there must be some  $T_0 \in (a, b)$  for which 0 is an eigenvalue. Hence Theorem 4.1 holds with the metric  $\overline{g}_{T_0}$  and the corresponding operator  $\tilde{D} = \tilde{D}_{T_0}$ .  $\square$

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